COUNTER EXAMPLE TO CANTOR'S THEOREM

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Abstract. Cantor's theorem on power set inequality state bijection of any arbitrary set X onto $P(X)$ does not exist and $|X| < |P(X)|$ which is disproved by considering Bijection between binary numbers(B) and Power set of binary numbers($P(B)$) proving $|B| = |P(B)|$ which furthur gives possible solution Continuum Hypothesis and related open questions are given.

In this paper only binary arithmatic is considered; B is set of Binary numbers and its power set is $P(B)$;Binary numbers are equivalent to Natural numbers in decimal number system; set B can also be named as Natural numbers of Binary number system. Binary digits are represented as follows where $i \in B$ and $n_i \in \{0,1\}$

$$\begin{aligned}\sum_{i=0}^{\infty} 10^i n_i &= \cdots n_{100}(10000) + n_{11}(1000) + n_{10}(100) + n_1(10) + n_0 \\ &= \cdots n_{100}(10)^{100} + n_{11}(10)^{11} + n_{10}(10)^{10} + n_1(10) + n_0\end{aligned}$$

1. Bijection between set of Binary numbers(B) and Power set of Binary numbers($P(B)$)

$f : B \to P(B)$

$$f(Y) = Z \begin{cases} \text{if } n_i = 1 \text{ then } i \in Z \\ \text{else } i \notin Z \end{cases}$$

OR

$$f(Y) = Z \begin{cases} \text{if } x \mod 10^i - x \mod 10^{i-1} = 1 \text{ then } i \in Z \\ \text{else } i \notin Z \end{cases}$$

$$f^{-1}(Z) = \sum_{i=0}^{\infty} 10^i n_i \begin{cases} \text{if } i \in Z \text{ then } n_i = 1 \\ \text{else } n_i = 0 \end{cases}$$

Proof for bijection. (a)Let $x, y \in B$ and $x \neq y$,

$$x = \sum_{i=0}^{\infty} 10^i a_i \qquad f(x) = X_0 \begin{cases} \text{if } a_i = 1 \text{ then } i \in Z \\ \text{else } i \notin Z \end{cases} \qquad a_i \in \{0,1\}$$

$$y = \sum_{i=0}^{\infty} 10^i b_i \qquad f(y) = Y_0 \begin{cases} \text{if } b_i = 1 \text{ then } i \in Z \\ \text{else } i \notin Z \end{cases} \qquad b_i \in \{0,1\}$$

since $x \neq y$ there exists atleast one i implying $a_i \neq b_i$ therefore for atleast one element $(i \in f(x) \wedge i \notin f(y))$ or $(i \notin f(x) \wedge i \in f(y))$ therefore $f(x) \neq f(y)$.

$$x \neq y \implies \exists i, a_i \neq b_i \implies (i \in f(x) \wedge i \notin f(y)) \vee (i \notin f(x) \wedge i \in f(y))$$

$$\implies f(x) \neq f(y)$$

TABLE 1. Mapping by function $f : B \to P(B)$

Y	$f(Y)$
001011	{1,10,100}
10000000	{111}
$\cdots 01010101$	$\{1, 11, 101, 111 \cdots\}$
0	{}
$\cdots 11111111$	B
$\vdots$	$\vdots$

(b)Let $K \in P(B)$ then $\exists f^{-1}(K) \in B$

$$f^{-1}(K) = \sum_{i=0}^{\infty} 10^i n_i \begin{cases} \text{if } i \in K \text{ then } n_i = 1 \\ \text{else } n_i = 0 \end{cases} \tag{1}$$

$i \in B$ and $n_i \in \{0, 1\}$

$$f(f^{-1}(K)) = Q \begin{cases} \text{if } n_i = 1 \text{ then } i \in Q \\ \text{else } i \notin Q \end{cases}$$

$$\text{from equation1 value of } f^{-1}(K) \implies (n_i = 1 \iff i \in P)$$

$$f(f^{-1}(K)) = Q \begin{cases} \text{if } i \in K \text{ then } i \in Q \\ \text{else } i \notin Q \end{cases}$$

$$f(f^{-1}(K)) = Q = Z$$

from (a) and (b)

$$(i)\forall Z \in P(B), \exists f^{-1}(Z) \in B \implies f(f^{-1}(Z)) = Z$$

$$(ii)\forall x, y \in B \, [(x = y) \iff f(x) = f(y)]$$

therefore $f : B \to P(B)$ is proved bijection. ☐

IMPLICATIONS

Theorem1.1 and Definition1.1 are prerequisite to discuss implication

Theorem 1.1 (Cantor's Theorem)**.** [3] $|X| < |P(X)|$, *there is not bijection between X and $P(X)$; where X is any set.*

Definition 1.1. *Aleph hierarchy*[1] *is given as follows Cardinality of Natural numbers*$=\aleph_0 < \aleph_1 < \aleph_2 < \cdots$ [2]

Theorem 1.2. *Any arbitary set X satisfies $|X| \leq |P(X)|$*

Proof. Considering bijection between B and $P(B)$ (section 1) proved $|B| = |P(B)|$ and disproving Cantor's theorem 1.1. ☐

Theorem 1.3. $\aleph_0 = 2^{\aleph_0}$

Proof. Set B is eqvivalent to Decimal Natural numbers therefore

$$(|B| = |P(B)|) \implies \aleph_0 = 2^{\aleph_0}$$

□

Continuum Hypothesis(CH). [1] Continuum Hypothesis states $2^{\aleph_0} = \aleph_1$ by theorem 1.3 which can be written as $\aleph_0 = \aleph_1$ clearly it is contradictory to definition of Cardinal numbers(aleph hierarchy) 1.1.

Two possible solutions can be stated as follows:

(1) If CH true and Aleph hierarchy is false;

$$\aleph_0 = \aleph_1$$

(2) else CH is false and Aleph hierarchy is true;

$$2^{\aleph_0} < \aleph_1$$

OPEN QUESTIONS

- Can transfinite set thoery and Cantor's Argument be proved false? since logical deduction cantor used are based on inequality $|X| < |P(X)|$ which is proved false.
- Does every infinite set X satisfy $|X| = |P(X)|$,i.e. for all k ($\aleph_k = 2^{\aleph_k}$)?
- Solution to Continuum Hypothesis and its Implications to cardinal numbers.
- Bijection between Real numbers and Natural numbers OR proof for existence or not existence of bijection between real numbers and natural numbers.

REFERENCES

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2. Poonen, B. Infinity: Cardinal Numbers. (2002)
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